

Fibonacci lattice points

JAVIER CILLERUELO

Instituto de Ciencias Matemáticas (CSIC-UAM-UC3M-UCM) and
Departamento de Matemáticas
Universidad Autónoma de Madrid
28049, Madrid, España
franciscojavier.cilleruelo@uam.es

FLORIAN LUCA

Instituto de Matemáticas
Universidad Nacional Autonoma de México
C.P. 58089, Morelia, Michoacán, México
fluca@matmor.unam.mx

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1 Introduction

Let $r(n)$ be the number of integer solutions (x, y) of the Diophantine equation $x^2 + y^2 = n$. It is known that $r(n)$ is an unbounded function. Consider the polygon with vertices in the $r(n)$ lattice points (x, y) . Clearly, all these lattice points lie on the circle of radius \sqrt{n} centered in the origin. The distribution of these points on the above circle was studied in [3] and [4]. In order to study the above distribution, let $S(n)$ denotes the area of the polygon whose vertices are the $r(n)$ lattice points. If the above $r(n)$ lattice points are well-distributed, then $S(n)$ should be close to the area of the circle which is πn .

If $r(n) > 0$, then trivially $2/\pi \leq S(n)/\pi n < 1$. In [3], it was proved that the inequality $|S(n)/\pi n - 1| \ll (\log \log n / \log n)^2$ holds infinitely often. In [4], it was shown that the inequality $|S(n)/\pi n - 1| \ll (\log \log \log n / \log \log n)^2$ holds for most positive integers n having $r(n) > 0$.

Let $(F_n)_{n \geq 0}$ be the Fibonacci sequence given by $F_0 = 0$, $F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 0$. Since

$$F_{2n+1} = F_n^2 + F_{n+1}^2, \tag{1}$$

it follows that $r(F_{2n+1}) > 0$ for all $n \geq 0$. In [6] and Lemma 1 in [1], it was shown that the equality $r(F_{2n}) = 0$ holds for most positive integers n .

In this paper, we investigate the size of $S(F_{2n+1})$. We have the following result.

Theorem 1. *There exists a positive constant c_1 such that*

(i) *The inequality*

$$\left| \frac{S(F_n)}{\pi F_n} - 1 \right| \ll \frac{1}{(\log \log n)^{c_1}} \quad (2)$$

holds for most odd integers n .

(ii) *The inequality*

$$\left| \frac{S(F_n)}{\pi F_n} - 1 \right| \ll \left(\frac{\log \log n}{\log n} \right)^{c_1} \quad (3)$$

holds for infinitely many positive integers n .

In [3], it was also proved that the set $\{S(n)/\pi n : r(n) > 0\}$ is dense in $[2/\pi, 1]$. It turns out that this is not the case for the set $\{S(F_{2n+1})/\pi F_{2n+1}\}_{n \geq 0}$. We have the following result.

Theorem 2. *i) For any $\epsilon > 0$, the elements of the set $\{S(F_{2n+1})/\pi F_{2n+1}\}_{n \geq 0}$ lying in $[\frac{2}{\pi}, \frac{6}{\pi\sqrt{5}} - \epsilon]$ form a finite set.*

ii) The number of elements of the set $\{S(F_{2n+1})/\pi F_{2n+1}\}_{n \geq 0}$ lying in $[\frac{2}{\pi}, \frac{6}{\pi\sqrt{5}}]$ is infinite if and only if the sequence F_{4n+3} contains infinitely primes.

We believe that the set $\{S(F_{2n+1})/\pi F_{2n+1}\}_{n \geq 0} \cap [\frac{6}{\pi\sqrt{5}}, 1]$ is a dense set in $[\frac{6}{\pi\sqrt{5}}, 1]$, but this seems to be a very difficult problem to solve.

Throughout this paper, we use the standard notations \ll , \gg , O and o with their regular meaning.

2 The Proof of Theorem 1

We start with (i). Let x be a large positive real number. Let $n \leq x$ be odd. We show that estimate (2) holds for all such n with $o(x)$ exceptions as $x \rightarrow \infty$. We may assume that $n \geq x/\log x$. Let $\omega(n)$ be the number of distinct prime factors of n . By the Turán-Kubilius estimate,

$$\sum_{n \leq x} (\omega(n) - \log \log x)^2 = O(x \log \log x),$$

it follows that the estimate $\omega(n) > 0.5 \log \log x$ holds for all odd $n \leq x$ with $o(x)$ exceptions as $x \rightarrow \infty$. Since $\pi(10 \log \log x) \ll \log \log x / \log \log \log x = o(\log \log x)$, it follows that for all odd $n \leq x$ have at least $K := \lfloor 0.25 \log \log x \rfloor$ distinct prime factors $p > L := \lfloor 10 \log \log x \rfloor$ except for some subset of them of cardinality $o(x)$ as $x \rightarrow \infty$. Write

$$n = p_1 \cdots p_K m,$$

where $L < p_1 < \cdots < p_K$ are distinct primes and m is an integer. Since $p_i \mid n$, it follows that $F_{p_i} \mid F_n$ for $i = 1, \dots, K$. Since F_a and F_b are coprime when a and b are coprime positive integers, it follows that $\prod_{i=1}^K F_{p_i}$ is a divisor of F_n . Write

$$F_n = F_{p_1} \cdots F_{p_K} M,$$

for some positive integer M . Since n is odd, it follows that all divisors of F_n are either 2 or are primes which are congruent to 1 modulo 4. Indeed, this is an easy consequence of representation (1) together with the fact that F_n and F_{n+1} are coprime. Now by (1) we have

$$F_{p_i} = F_{(p_i-1)/2}^2 + F_{(p_i+1)/2}^2.$$

Put

$$\Phi_{p_i} := (4/\pi) \tan^{-1} (F_{(p_i+1)/2} / F_{(p_i-1)/2}) \quad \text{for } i = 1, \dots, K.$$

Write also

$$F_m = \frac{\alpha^m - \beta^m}{\alpha - \beta} \quad \text{for } m \geq 0, \quad \text{where } (\alpha, \beta) = \left(\frac{1 + \sqrt{5}}{2}, \frac{1 - \sqrt{5}}{2} \right).$$

Then

$$\frac{F_{m+1}}{F_m} = \frac{\alpha^{m+1} - \beta^{m+1}}{\alpha^m - \beta^m} = \alpha + O\left(\frac{1}{\alpha^{2m}}\right)$$

holds for all $m \geq 1$, therefore

$$\frac{F_{(p_i+1)/2}}{F_{(p_i-1)/2}} = \alpha + O\left(\frac{1}{\alpha^{p_i}}\right) = \alpha + O\left(\frac{1}{(\log n)^2}\right) \quad \text{holds for } i = 1, \dots, K,$$

where we used the fact that since $p_i > L$, we have that $\alpha^{p_i} > (\log n)^2$. Thus, by Taylor's formula,

$$\Phi_{p_i} = \gamma + O\left(\frac{1}{(\log n)^2}\right) \quad \text{holds for } i = 1, \dots, K, \quad (4)$$

where $\gamma := (4/\pi)\tan^{-1}(\alpha)$. It is well-known and easy to prove that γ is irrational. Write also $M = a^2 + b^2$ with some integers $0 \leq a \leq b$ and put $\Phi = (4/\pi)\tan^{-1}(a/b)$. Proposition 2.4 in [4] implies that the angles

$$(\pi/4) \left(\Phi + \sum_{j=1}^K \varepsilon_j \Phi_{p_j} \right), \quad \varepsilon_j \in \{\pm 1\} \quad \text{for } j = 1, \dots, K \quad (5)$$

correspond to lattice points on the circle $x^2 + y^2 = F_n^2$. Using estimate (4) in (5), we get that among the angles (5) we also have all the angles

$$(\pi/4) \left(\Phi + (K - 2i)\gamma + O\left(\frac{\log \log n}{(\log n)^2}\right) \right), \quad i = 0, 1, \dots, K. \quad (6)$$

Indeed, this can be deduced by taking in (4) $K - i$ of the signs ε_j to equal 1 and the remaining i of them to equal -1 . We next show that for some constant $c_2 > 0$ every arc of length $> 1/K^{c_2}$ contains one of the points from list (6). To see why, take $\gamma_1 := \gamma/4$ and put $x_i = \{i\gamma_1\}$ for $i = 1, \dots, K$, where for a number x we write $\{x\}$ for the fractionary part of x . For each interval \mathcal{J} of $[0, 1)$, let $V(\mathcal{J}, K) = \{1 \leq i \leq K : x_i \in \mathcal{J}\}$. Let $D(K)$ be the discrepancy of the sequence $\mathbf{x} := (x_i)_{i=1}^K$ defined as

$$D(K) = \sup_{\mathcal{J} \subset [0, 1]} \left| \frac{V(\mathcal{J}, K)}{K} - |\mathcal{J}| \right|.$$

Here, $|\mathcal{J}|$ denotes the length of \mathcal{J} . By Theorem 3.2 in Chapter 2 in [5], we know that if the type of γ_1 is finite τ , then $D(K) \leq K^{-1/\tau+o(1)}$ as $K \rightarrow \infty$, where

$$\tau = \sup \left\{ \rho \in \mathbb{R} : \liminf_{m \rightarrow \infty} m^\rho \|\gamma_1 m\| = 0 \right\}.$$

Thus, assuming that τ is finite, it follows that if we take $c_2 := 1/(2\tau)$, and keeping in mind that the error in (6) tends to zero much faster than any power of negative exponent of K , we conclude that for large x , any arc of length $> 1/K^{c_2}$ contains one of the points from list (6).

It remains to justify that τ is finite. For this, it suffices to show that the inequality

$$|2\tan^{-1}(\alpha) - 2p\pi/q| \gg q^{-c_3}$$

holds for all rational numbers p/q with some constant c_3 . We may assume that $2p\pi/q$ is very close to $2\tan^{-1}(\alpha)$. Since $\tan(2\tan^{-1}(\alpha)) = 2$, we get

that

$$\begin{aligned}
|2 \tan^{-1}(\alpha) - 2p\pi/q| &\asymp |e^{i2 \tan^{-1}(\alpha)} - e^{2\pi i p/q}| \\
&= |e^{i \tan^{-1}(2)} - e^{2\pi i p/q}| \\
&= \left| \frac{1+2i}{\sqrt{5}} - e^{2\pi i p/q} \right| \\
&\gg \frac{1}{q} \left| \left(\frac{1+2i}{\sqrt{5}} \right)^q - 1 \right| \gg \frac{1}{q^{c_3}}
\end{aligned}$$

where all the above inequalities are obvious for $2p\pi/q$ in a small neighborhood $2 \tan^{-1}(\alpha)$ except for the last one which follows by a classical application of a lower bound for a linear form in logarithms of algebraic numbers.

Finally, an easy geometrical argument (see the proof of Proposition 3.1 in [4]) now shows that

$$\left| \frac{S(F_n)}{\pi F_n} - 1 \right| \ll \frac{1}{K^{2c_2}} \ll \frac{1}{(\log \log n)^{c_1}},$$

with $c_1 := 2c_2$, which is what we wanted to prove. This takes care of (i).

The proof of (ii) is similar, except that for (ii) we start with a large x , put $L := \lfloor 10 \log \log x \rfloor$ and let $p_1 < \dots < p_K$ be all the primes in $[L, (\log x)/2]$. By the Prime Number Theorem, $K \asymp \log x / \log \log x$. Put

$$n = p_1 \cdots p_K.$$

Again by the Prime Number Theorem, we have

$$n = \prod_{L \leq p \leq (\log x)/2} p = x^{1/2+o(1)}$$

as $x \rightarrow \infty$. We thus have that for large x the number n is odd and smaller than x . Now the previous argument shows that

$$\left| \frac{S(F_n)}{\pi F_n} - 1 \right| \ll \frac{1}{K^{2c_2}} \ll \left(\frac{\log \log x}{\log x} \right)^{c_1} \ll \left(\frac{\log \log n}{\log n} \right)^{c_1},$$

which is what we wanted to prove.

3 The proof of Theorem 2

Let us say that m is a 2-prime if $m = 2^k p$, where $k \geq 0$ and $p = 1$, or p is prime. Positive integers m which are 2-primes are characterized by the fact

that whenever $m = a^2 + b^2$ with integers a and b , then a and b are uniquely determined up to signs and order. We next record that if F_n is a 2-prime, then $n \in \{1, 2, 3, 4, 6, 8, 9\}$, or $n \geq 5$ is prime and F_n is also prime. Indeed, this follows easily from [2].

The proof of Theorem 2 is a corollary of the following lemma.

Lemma 1. *For any $n \geq 1$, then $\frac{S(F_{2n+1})}{\pi F_{2n+1}} < \frac{6}{\pi\sqrt{5}}$ if and only if n is odd and F_{2n+1} is a 2-prime.*

Proof. Since $F_{2n+1} = F_{n+1}^2 + F_n^2$, the circle $x^2 + y^2 = F_{2n+1}$ contains lattice points at the angles

$$\Psi_n, \pi/2 - \Psi_n, \Psi_n + \pi/2, \pi - \Psi_n, \Psi_n + \pi, 3\pi/2 - \Psi_n, \Psi_n + 3\pi/2, 2\pi - \Psi_n, \quad (7)$$

where $\Psi_n = \tan^{-1}(F_n/F_{n+1})$.

We observe that $\lim_{n \rightarrow \infty} \Psi_n = \Psi = \tan^{-1}(\alpha^{-1})$, that $\Psi_n < \Psi$ when n is even, and that $\Psi_n > \Psi$ if n is odd.

If ϕ_1, \dots, ϕ_k denote the counter-clockwise ordered angles of the lattice points on the circle of radius \sqrt{n} , we then have

$$S(n) = \frac{n}{2} \sum_{i=1}^k \sin(\phi_{i+1} - \phi_i), \quad (8)$$

where we make the convention that $\phi_{k+1} = \phi_1$.

In particular, the area of the polygon determined by the angles shown at (7) is

$$2F_{2n+1} (\cos(2\Psi_n) + \sin(2\Psi_n)).$$

Thus,

$$\frac{S(F_{2n+1})}{\pi F_{2n+1}} \geq \frac{2}{\pi} (\cos(2\Psi_n) + \sin(2\Psi_n)),$$

and the equality holds if the circle $x^2 + y^2 = F_{2n+1}$ does not contain more angles than the (at most) eight angles described above, and this happens only if F_{2n+1} is a 2-prime.

Since $\tan^{-1}(1/2) \leq \Psi_n \leq \tan^{-1}(1)$ for $n \geq 1$, and the function $f(x) = \cos(2x) + \sin(2x)$ is decreasing in that interval, we deduce that:

- If n is even, then $\frac{S(F_{2n+1})}{\pi F_{2n+1}} > \frac{2}{\pi} f(\Psi) = \frac{6}{\pi\sqrt{5}}$;
- If n is odd and F_{2n+1} is a 2-prime, then $\frac{S(F_{2n+1})}{\pi F_{2n+1}} < \frac{2}{\pi} f(\Psi) = \frac{6}{\pi\sqrt{5}}$.

To conclude the proof, we have to prove that if n is odd and F_{2n+1} is not a 2-prime, then $S(F_{2n+1})/\pi F_{2n+1} > \frac{6}{\pi\sqrt{5}}$. Since $F_3 = 2, F_7 = 13$ and $F_{11} = 89$ are prime numbers, we can assume that $n \geq 7$.

If F_{2n+1} is not prime and n is odd, then besides Ψ_n , which is larger than Ψ when n is odd, there exist other lattice points on the circle $x^2 + y^2 = F_{2n+1}$ with an angle $\Phi \in [0, \pi/4]$.

Thus, the circle $x^2 + y^2 = F_{2n+1}$ contains lattice points at angles $\Psi_n, \Phi, \frac{\pi}{2} - \Phi, \frac{\pi}{2} - \Psi_n$ and all the translations of these angles by a multiple of $\pi/2$.

Using (8), we can compute that the area of the polygon determined by these angles equals:

$$\begin{cases} \frac{F_{2n+1}}{2} (8 \sin(\Phi - \Psi_n) + 4 \cos(2\Phi) + 4 \cos(2\Psi_n)), & \text{if } \Phi > \Psi_n; \\ \frac{F_{2n+1}}{2} (8 \sin(\Psi_n - \Phi) + 4 \cos(2\Psi_n) + 4 \cos(2\Phi)), & \text{if } \Phi < \Psi_n. \end{cases}$$

Using easy trigonometric manipulations, we can resume both formulas above in the common expression

$$2F_{2n+1}(2 \sin(|\Psi_n - \Phi|)(1 - \sin(\Phi + \Psi_n)) + f(\Psi_n)),$$

where $f(x) = \cos(2x) + \sin(2x)$. Thus,

$$\frac{S(F_{2n+1})}{2F_{2n+1}} \geq 2 \sin(|\Psi_n - \Phi|)(1 - \sin(\Phi + \Psi_n)) + f(\Psi_n).$$

Since the distance between two lattice points in the circle is $\geq \sqrt{2}$, we have that

$$|\Phi - \Psi_n| > \sqrt{2}/\sqrt{F_{2n+1}}. \quad (9)$$

On the other hand, it is easy to compute that for n odd we have

$$\tan \Psi_n - \tan \Psi = \frac{F_n}{F_{n+1}} - \alpha^{-1} = \frac{\sqrt{5}}{\alpha^{2n+2} - 1}. \quad (10)$$

We have to prove that

$$2 \sin(|\Phi - \Psi_n|)(1 - \sin(\Phi + \Psi_n)) + f(\Psi_n) \geq f(\Psi) = \frac{3}{\sqrt{5}},$$

which is equivalent to proving that

$$2 \sin(|\Phi - \Psi_n|)(1 - \sin(\Phi + \Psi_n)) > f(\Psi) - f(\Psi_n).$$

To estimate the left hand from below, we will use that $\sin x \geq \frac{2\sqrt{2}}{\pi}x$ for $0 < x \leq \pi/4$, that $\Phi + \Psi_n \leq \pi/4 + \tan^{-1}(13/21)$, when $n \geq 7$ is odd, and that the estimate $|\Phi - \Psi_n| \geq \sqrt{2}/\sqrt{F_{2n+1}}$ holds. Thus,

$$\begin{aligned} 2 \sin(\Phi - \Psi_n) (1 - \sin(\Phi + \Psi_n)) &\geq \frac{4\sqrt{2}}{\pi}(\Phi - \Psi_n) \left(1 - \sin\left(\frac{\pi}{4} + \tan^{-1}\left(\frac{13}{21}\right)\right)\right) \\ &\geq \frac{8}{\pi} \left(1 - \frac{34}{\sqrt{610}}\right) \frac{1}{\sqrt{F_{2n+1}}}. \end{aligned}$$

To estimate the right hand from above, we use that for $n \geq 7$ odd, we have that $\alpha^{-1} < \tan(\Psi_n) \leq 13/21$, and that $F_{2n+2} < \alpha^{2n+2}/\sqrt{5}$. Hence,

$$\begin{aligned} f(\Psi) - f(\Psi_n) &= \cos(2\Psi) - \cos(2\Psi_n) + \sin(2\Psi) - \sin(2\Psi_n) \\ &= 2 \left(\frac{1}{\tan^2(\Psi) + 1} - \frac{1}{\tan^2(\Psi_n) + 1} + \frac{\tan(\Psi)}{\tan^2(\Psi) + 1} - \frac{\tan(\Psi_n)}{\tan^2(\Psi_n) + 1} \right) \\ &= (\tan(\Psi_n) - \tan(\Psi)) \left(\frac{\tan(\Psi) \tan(\Psi_n) + \tan(\Psi) + \tan(\Psi_n) - 1}{(\tan^2(\Psi) + 1)(\tan^2(\Psi_n) + 1)} \right) \\ &\leq (\tan(\Psi_n) - \tan(\Psi)) \left(\frac{\tan(\Psi) \tan(\Psi_n) + \tan(\Psi) + \tan(\Psi_n) - 1}{(\tan^2(\Psi) + 1)^2} \right) \\ &= (\tan(\Psi_n) - \tan(\Psi)) \left(\frac{\alpha^{-1} \tan(\Psi_n) + \alpha^{-1} + \tan(\Psi_n) - 1}{(\alpha^{-2} + 1)^2} \right) \\ &= (\tan(\Psi_n) - \tan(\Psi)) \left(\frac{(2 + \sqrt{5}) \tan(\Psi_n) - 1}{5} \right) \\ &< \frac{(2 + \sqrt{5}) \tan(\Psi_n) - 1}{\sqrt{5}(\alpha^{2n+2} - 1)} < \frac{(2 + \sqrt{5})(13/21) - 1}{\sqrt{5}(\sqrt{5}F_{2n+2} - 1)} \\ &< \left(\frac{5 + 13\sqrt{5}}{105} \right) \frac{1}{F_{2n+2} - 1}. \end{aligned}$$

Finally, we observe that the inequality

$$\frac{8}{\pi} \left(1 - \frac{34}{\sqrt{610}}\right) \frac{1}{\sqrt{F_{2n+1}}} > \left(\frac{5 + 13\sqrt{5}}{105} \right) \frac{1}{F_{2n+2} - 1}$$

holds for $n \geq 7$. This completes the proof of the lemma. \square

To derive Theorem 2 from the lemma above, we observe that only when F_{2n+1} is 2-prime and n is odd (hence, F_{2n+1} is prime), we have that

$$\frac{S(F_{2n+1})}{\pi F_{4n-1}} < \frac{6}{\pi\sqrt{5}},$$

and that

$$\lim_{\substack{n \rightarrow \infty \\ F_{2n+1} \text{ 2-prime}}} \frac{S(F_{2n+1})}{\pi F_{2n+1}} = \frac{6}{\pi\sqrt{5}}.$$

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